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SUBMANIFOLDS IN DE SITTER SPACE-TIME SATISFYING $\Delta H = \lambda H$

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ABSTRACT

In [3] the author initiated the study of submanifolds whose mean curvature vector H is an eigenvector of the Laplacian Δ and proved that such submanifolds are either biharmonic or of 1-type or of null 2-type. The classification of surfaces with $\Delta H = \lambda H$ in a Euclidean 3-space was done by the author in 1988. Moreover, in [4] the author classified such submanifolds in hyperbolic spaces. In this article we study this problem for space-like submanifolds of the Minkowski space-time E_1^m when the submanifolds lie in a de Sitter space-time. As a result, we characterize and classify such submanifolds in de Sitter space-times.

1. Introduction

Let E_1^m be the m-dimensional Minkowski space-time with the standard flat metric given by

$$g = -dt^2 + \sum_{j=2}^{m} dx_j^2, \tag{1.1}$$

where $(t, x_2, \ldots, x_m)$ is a rectangular coordinate system of E_1^m. For a positive number r and a point $c \in E_1^m$, we denote by $S_1^{m-1}(c, r)$ and $H^{m-1}(c, -r)$ the de Sitter space-time and the hyperbolic space defined respectively by

$$S_1^{m-1}(c, r) = \{x \in E_1^m \colon \langle x - c, x - c \rangle = r^2\}, \tag{1.2}$$

$$H^{m-1}(c, -r) = \{x \in E_1^m \colon < x - c, x - c >= -r^2 \text{ and } t > 0\}, \tag{1.3}$$

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where < , > denotes the indefinite inner product on the Minkowski space-time E_1^m. The point c is called the center of $S_1^{m-1}(c,r)$ and of $H^{m-1}(c,-r)$, respectively. We simply denote $S_1^{m-1}(0,r)$ and $H^{m-1}(0,-r)$ by $S_1^{m-1}(r)$ and $H^{m-1}(-r)$, respectively.

Let x: $M \to E_1^m$ be an isometric immersion from an n-dimensional Riemannian manifold M into E_1^m. Denote the position vector field of the immersion x: $M \to E_1^m$ also by x. Then we have (cf. [2])

$$\Delta x = -nH, \tag{1.4}$$

where H is the mean curvature vector of M in E_1^m.

If M is immersed into the de Sitter space-time $S_1^{m-1}(1)$ as a minimal submanifold, then $H = -x$ (cf. [2]). Thus, (1.4) yields $\Delta H = nH$. This shows that the mean curvature vector H of M in the Minkowski space-time E_1^m is an eigenvector of the Laplacian Δ of M with eigenvalue n. The classification of surfaces with $\Delta H = \lambda H$ for some constant λ in a Euclidean 3-space was done by the author in 1988. In [4] the author classified submanifolds of the hyperbolic space $H^{m-1}(-1) \subset E_1^m$ whose mean curvature vector in E_1^m is an eigenvector of Δ.

In this paper, we study isometric immersions x: $M \to S_1^{m-1}(1) \subset E_1^m$ whose mean curvature vector H in E_1^m is an eigenvector of Δ, i.e., $\Delta H = \lambda H$ for some constant λ. Several classification theorems in this resepct are obtained.

2. Preliminaries

Let x: $M \to E_1^m$ be an isometric immersion of an n-dimensional, Riemannian manifold M into the Minkowski space-time E_1^m. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on M and E_1^m, respectively. Let A, H, D and σ be the Weingarten map, the mean curvature vector, the normal connection, and the second fundamental form of M in E_1^m, respectively. We choose orthonormal local frame fields $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ on M such that $e_1, \ldots, e_n$ are tangent to M and $e_{n+1}, \ldots, e_m$ are normal to M. Put $\epsilon_r = \langle e_r, e_r \rangle$, $r = 1, \ldots, m$. Then we have the following useful formula first obtained in [1, 2]:

$$\Delta H = \Delta^D H + \frac{n}{2} \operatorname{grad} \langle H, H \rangle + 2 \operatorname{trace} A_{DH} + \sum_{r=n+1}^{m} \epsilon_r \operatorname{trace}(A_H A_r) e_r, \tag{2.1}$$

where Δ^D is the Laplacian of the normal bundle and $A_r = A_{e_r}, r = n+1, \ldots, m$.

If M lies in the de Sitter space-time $S_1^{m-1}(1)$, then the position vector x is a unit normal vector field of M which is also normal to $S_1^{m-1}(1)$. If we choose $e_{n+1} = x$ and let H' denote the mean curvature vector of M in $S_1^{m-1}(1)$, then we have

$$H = H' - x. \tag{2.2}$$

From (2.1) and (2.2) we find

$$\begin{aligned}\Delta H =& \Delta^D H + \frac{n}{2}\operatorname{grad}\langle H, H\rangle + 2\operatorname{trace} A_{DH} + nH' \\ &- n(\langle H', H'\rangle + 1)x + \sum_{r=n+2}^{m} \epsilon_r \operatorname{trace}(A_{H'}A_r)e_r.\end{aligned} \tag{2.3}$$

Since $e_{n+1} = x$, one has $\tilde{\nabla}_X e_{n+1} = X = -A_{n+1}X + D_X e_{n+1}$. Thus

$$A_{n+1} = A_x = -I, \quad De_{n+1} = 0. \tag{2.4}$$

We put

$$De_r = \sum_{t=n+1}^{m} \omega_r^t e_t. \tag{2.5}$$

Then we have

$$\omega_r^s = -\epsilon_r \epsilon_s \omega_s^r, \quad r, s, t = n+1, \ldots, m. \tag{2.6}$$

Moreover, from (2.4), we have

$$\omega_{n+1}^{n+2} = \cdots = \omega_{n+1}^{m} = 0. \tag{2.7}$$

By using (2.5), (2.6) and (2.7) and by a direct computation we may obtain

$$\begin{aligned}\Delta^D e_{n+2} =& -\sum_{r=n+3}^{m} \left\{ \sum_{i=1}^{n} \sum_{t=n+3}^{m} \omega_{n+2}^t(e_i)\omega_t^r(e_i) + \operatorname{trace}(\nabla \omega_{n+2}^r) \right\} e_r \\ &- \sum_{t=n+3}^{m} \sum_{i=1}^{n} \omega_{n+2}^t(e_i)\omega_t^{n+2}(e_i)e_{n+2},\end{aligned} \tag{2.8}$$

where

$$\operatorname{trace}(\nabla \omega_{n+2}^r) = \sum_{i=1}^{n} (\nabla_{e_i}\omega_{n+2}^r)(e_i).$$

We recall the following result.

PROPOSITION 1: *Let M be a pseudo-Riemannian submanifold of a pseudo-Euclidean space E_s^m. Then the mean curvature vector H of M in E_s^m satisfies $\Delta H = \lambda H$ for some constant λ if and only if either M is a biharmonic submanifold of E_s^m, i.e., $\Delta H = 0$, or M is of 1-type or of null 2-type.*

Remark 1: Although Proposition 1 was stated in [3] only for submanifolds in Euclidean space, it is true for every pseudo-Riemannian submanifold of a pseudo-Euclidean space since the exact proof given in [3] works for pseudo-Riemannian submanifolds as well. (For general information on submanifolds of finite type, see, for instance, [1, 7].)

3. Submanifolds satisfying $\Delta H = \lambda H$ with $\lambda < n$

In this section we completely classify space-like submanifolds of $S_1^{m-1}(1)$ satisfying $\Delta H = \lambda H$ with $\lambda < n$.

THEOREM 2: *Let M be an n-dimensional space-like submanifold of the de Sitter space-time $S_1^{m-1}(1)$, imbedded standardly in the Minkowski space-time E_1^m. Then the mean curvature vector H of M in E_1^m satisfies $\Delta H = \lambda H$ for some $\lambda < n$ if and only if M is contained in a space-like, non-totally geodesic, totally umiblical hypersurface of $S_1^{m-1}(1)$ as a minimal submanifold.*

Proof: Let M be an n-dimensional space-like submanifold of the de Sitter space-time $S_1^{m-1}(1)$ which is imbedded standardly in the Minkowski space-time E_1^m. Assume the mean curvature vector H of M in E_1^m satisfies $\Delta H = \lambda H$ for some $\lambda < n$. Then, by (2.2) and (2.3), we find

$$<H', H'> = \frac{\lambda}{n} - 1 < 0, \quad <H, H> = \frac{\lambda}{n}. \tag{3.1}$$

Thus, M has constant mean curvatures in $S_1^{m-1}(-1)$ and in E_1^m. We put

$$H' = a e_{n+2}, \quad a^2 = 1 - \frac{\lambda}{n}. \tag{3.2}$$

We have

$$\epsilon_{n+2} = -1, \quad \epsilon_{n+3} = \cdots = \epsilon_m = 1. \tag{3.3}$$

From (2.8), (3.2) and (3.3) we find

(3.4)
$$\Delta^D H = \Delta^D H' = a \sum_{r=n+3}^{m} \left\{ \sum_{i=1}^{n} \sum_{t=n+3}^{m} \omega_{n+2}^t(e_i)\omega_r^t(e_i) - \operatorname{trace}(\nabla \omega_{n+2}^r) \right\} e_r$$
$$- ||De_{n+2}||^2 H',$$

where

$$(3.5)\qquad ||De_{n+2}||^2 = \sum_{i=1}^{n} \langle D_{e_i} e_{n+2}, D_{e_i} e_{n+2}\rangle \,.$$

By using (2.2), (2,3), (3.4) and $\Delta H = \lambda H$ we may obtain

$$(3.6)\qquad n - \operatorname{trace}(A_{n+2}^2) - ||De_{n+2}||^2 = \lambda.$$

Combining (3.1) and (3.6) we get

$$(3.7)\qquad n \operatorname{trace}(A_{n+2}^2) - (\operatorname{trace} A_{n+2})^2 = -n\, ||De_{n+2}||^2.$$

Let $\kappa_1, \ldots, \kappa_n$ be the eigenvalues of A_{n+2}. We have

$$(3.8)\qquad n \operatorname{trace}(A_{n+2}^2) - (\operatorname{trace}(A_{n+2}))^2 = \sum_{i<j} (\kappa_i - \kappa_j)^2.$$

From this we see that the left-hand-side of (3.7) is ≥ 0 and it is equal to 0 if and only if M is pseudo-umbilical in $S_1^{m-1}(1)$, i.e., $A_{H'} = \mu I$ for some nonzero function μ on M. On the other hand, because $e_{n+3}, \ldots, e_m$ are space-like, (3.5) yields $||De_{n+2}|| \geq 0$. Therefore, both sides of (3.7) vanish identically. Consequently, we have

$$(3.9)\qquad A_{H'} = \left(\frac{\lambda}{n} - 1\right) I, \quad A_H = \left(\frac{\lambda}{n}\right) I, \quad DH' = DH = 0.$$

Let

$$b = x - \frac{n}{\lambda} H.$$

Then

$$(3.10)\qquad < b, x >= 1 + \frac{n}{\lambda}.$$

Moreover, (3.9) implies that b is a constant nonzero vector in E_1^m; moreover, we also have $\langle x - b, x - b\rangle = n/\lambda$. Therefore, M also lies in the de Sitter space-time $S_1^{m-1}(b, \sqrt{n/\lambda})$. Let N be the intersection of $S_1^{m-1}(b, \sqrt{n/\lambda})$ and $S_1^{m-1}(1)$. Then N is a non-totally geodesic hypersurface of $S_1^{m-1}(1)$. Because

$$H = \frac{\lambda}{n}(x - b),$$

M is minimal in $S_1^{m-1}(b, \sqrt{n/\lambda})$; and hence M is also a minimal submanifold of the hypersurface N of $S_1^{m-1}(b, \sqrt{n/\lambda})$.

Let

$$\xi = \left(1 + \frac{n}{\lambda}\right) x - b.$$

Then, ξ is a time-like normal vector field of N in $S_1^{m-1}(1)$. From the definition of ξ, we see that the Weingarten map of N in $S_1^{m-1}(1)$ at ξ is given by $-(1+n/\lambda)I$. This shows that N is a space-like, totally umbilical hypersurface of $S_1^{m-1}(1)$ with nonzero constant mean curvature. Consequently, we have proved that if M satisfies the condition: $\Delta H = \lambda H$ for some constant $\lambda < n$, then M is contained in a space-like, non-totally geodesic, totally umbilical hypersurface of $S_1^{m-1}(1)$ as a minimal submanifold.

Conversely, if M is contained in a space-like, non-totally geodesic, totally umbilical hypersurface N of $S_1^{m-1}(1)$ as a minimal submanifold, then the mean curvature vector H' of M in $S_1^{m-1}(1)$ is given by a constant multiple of the unit normal vector field of N in $S_1^{m-1}(1)$, restricted to M. Thus, M is a pseudo-umbilical submanifold in $S_1^{m-1}(1)$ with nonzero parallel mean curvature vector. Therefore,

$$DH' = DH = 0 \quad \text{and} \quad A_{H'} = \mu I, \tag{3.11}$$

where μ is nonzero constant. From (3.11) we find

$$\Delta^D H = 0, \quad \langle H', H'\rangle = \mu, \quad \sum_{r=n+2}^{m} \epsilon_r \operatorname{trace}(A_{H'}A_r)e_r = n\mu H'. \tag{3.12}$$

Therefore, by (2.3), we get $\Delta H = nH' - n(\mu+1)x + n\mu H' = n(\mu+1)(H'-x) = n(\mu+1)H$. This shows that H is an eigenvector of Δ with eigenvalue $\lambda = n(\mu+1)$. Since H' is time-like, $\lambda < n$. ∎

If a submanifold M is contained in a totally umbilical hypersurface of $S_1^{m-1}(1)$ as a minimal submanifold, then M is a 1-type submanifold of E_1^m. Therefore, by applying Theorem 2, we obtain the following

COROLLARY 3: *If M is an n-dimensional, space-like submanifold of the de Sitter space-time $S_1^{m-1}(1)$ whose mean curvature vector H in E_1^m satisfies $\Delta H = \lambda H$ with $\lambda < n$, then M is of 1-type.*

4. Submanifolds satisfying $\Delta H = nH$

In this section we completely classify n-dimensional space-like submanifolds of $S_1^{m-1}(1)$ whose mean curvature vector in E_1^m is an eigenvector of Δ with eigenvalue n.

THEOREM 4: *Let $x\colon M \to S^{m-1}(1) \subset E_1^m$ be an isometric immersion from an n-dimensional Riemannian manifold M into the de Sitter space-time $S_1^{m-1}(1)$ which is imbedded standardly in the Minkowski space-time E_1^m. Then the mean curvature vector H of M in E_1^m is an eigenvector of Δ with eigenvalue n, i.e., $\Delta H = nH$, if and only if either* (a) *M is a minimal submanifold of $S_1^{m-1}(1)$ or* (b) *up to rigid motions of E_1^m, the isometric immersion x is given by*

$$x = (f + h, f + h, x_3, \ldots, x_m), \tag{4.1}$$

where h is a harmonic function, f is an eigenfunction of Δ with eigenvalue n, and $y = (x_3, \ldots, x_m)\colon M \to S^{m-3}(1) \subset E^{m-2}$ is an isometric minimal immersion from M into the unit hypersphere $S^{m-3}(1)$ of a Euclidean $(m-2)$-space E^{m-2}.

Proof: Let $x\colon M \to S^{m-1}(1) \subset E_1^m$ be an isometric immersion from an n-dimensional Riemannian manifold M into the de Sitter space-time $S_1^{m-1}(1)$ which is imbedded standardly in the Minkowski space-time E_1^m. Assume M satisfies the condition

$$\Delta H = nH. \tag{4.2}$$

Then, by (2.2), (2.3) and (4.2), we find

$$\langle H', H' \rangle = 0. \tag{4.3}$$

If $H' \equiv 0$, then M is a minimal submanifold of $S_1^{m-1}(1)$. Therefore, we assume $H' \neq 0$.

Since H' is a nonzero light-like vector field by (4.3), we may choose an orthonormal normal frame field $e_{n+1}, \ldots, e_m$ such that e_{n+2} is time-like, $e_{n+3}, \ldots, e_m$ are space-like, and moreover

$$H' = \delta(e_{n+2} + e_{n+3}), \quad \delta \neq 0. \tag{4.4}$$

From (2.2), (2.5), (2.6), (2.7) and (4.4) we find

$$\begin{aligned}\Delta^D H =&(\Delta\delta)(e_{n+2}+e_{n+3}) - 2\sum_{r=n+2}^{m}\{\omega_{n+2}^r(\nabla\delta)+\omega_{n+3}^r(\nabla\delta)\}e_r\\ &-\delta\sum_{r=n+2}^{m}\{\mathrm{trace}(\nabla\omega_{n+2}^r)+\mathrm{trace}(\nabla\omega_{n+3}^r)\}e_r\\ &-\delta\sum_{i=1}^{n}\sum_{r,t=n+2}^{m}\{\omega_{n+2}^t(e_i)+\omega_{n+3}^t(e_i)\}\omega_t^r(e_i)e_r.\end{aligned}\tag{4.5}$$

Combining this with (2.6), we find

$$\begin{aligned}\Delta^D H =&\left\{\left(\frac{\Delta\delta}{\delta}\right) - \mathrm{trace}(\nabla\omega_{n+3}^{n+2}) - 2\omega_{n+3}^{n+2}(\nabla\ln\delta)\right\}H'\\ &-\delta\sum_{i=1}^{n}\sum_{t=n+2}^{m}\{\omega_{n+2}^t(e_i)\omega_t^{n+2}(e_i)+\omega_{n+3}^t(e_i)\omega_t^{n+2}(e_i)\}e_{n+2}\\ &-\delta\sum_{i=1}^{n}\sum_{t=n+2}^{m}\{\omega_{n+2}^t(e_i)\omega_t^{n+3}(e_i)+\omega_{n+3}^t(e_i)\omega_t^{n+3}(e_i)\}e_{n+3}\\ -\delta\sum_{r=n+4}^{m}&\{\mathrm{trace}(\nabla\omega_{n+2}^r)+\mathrm{trace}(\nabla\omega_{n+3}^r)+2\omega_{n+2}^r(\nabla\ln\delta)+2\omega_{n+3}^r(\nabla\ln\delta)\\ &+\sum_{i=1}^{n}\sum_{t=n+2}^{m}(\omega_{n+2}^t(e_i)+\omega_{n+3}^t(e_i))\omega_t^r(e_i)\}e_r.\end{aligned}\tag{4.6}$$

Since $\Delta H = nH = nH' - nx$, (2.2), (2.3) and (4.6) imply

$$\begin{aligned}nH' =&\left\{\left(\frac{\Delta\delta}{\delta}\right) - \mathrm{trace}(\nabla\omega_{n+3}^{n+2}) - 2\omega_{n+3}^{n+2}(\nabla\ln\delta)+n\right\}H'\\ &-\delta\sum_{i=1}^{n}\sum_{t=n+2}^{m}\{\omega_{n+2}^t(e_i)\omega_t^{n+2}(e_i)+\omega_{n+3}^t(e_i)\omega_t^{n+2}(e_i)\}e_{n+2}\\ &-\delta\sum_{i=1}^{n}\sum_{t=n+2}^{m}\{\omega_{n+2}^t(e_i)\omega_t^{n+3}(e_i)+\omega_{n+3}^t(e_i)\omega_t^{n+3}(e_i)\}e_{n+3}\\ &-\mathrm{trace}(A_{H'}A_{n+2})e_{n+2}+\mathrm{trace}(A_{H'}A_{n+3})e_{n+3}.\end{aligned}$$

By taking the inner product of this formula with H' and applying $\epsilon_{n+2}=-1$, $\epsilon_{n+3}=1$, and (4.3) we find

$$\delta\sum_{i=1}^{n}\sum_{t=n+2}^{m}\{\omega_{n+2}^t(e_i)\omega_t^{n+2}(e_i)+\omega_{n+3}^t(e_i)\omega_t^{n+2}(e_i)\}+\mathrm{trace}(A_{H'}A_{n+2})$$

$$-\delta\sum_{i=1}^{n}\sum_{t=n+2}^{m}\{\omega_{n+2}^{t}(e_i)\omega_t^{n+3}(e_i)+\omega_{n+3}^{t}(e_i)\omega_t^{n+3}(e_i)\}+\mathrm{trace}(A_{H'}A_{n+3})=0.$$

From this, (2.6) and (4.4) we obtain

$$\begin{aligned}\sum_{i=1}^{n}\sum_{t=n+4}^{m}\{(\omega_{n+2}^{t}(e_i))^2+2\omega_{n+2}^{t}(e_i)\omega_{n+3}^{t}(e_i)+(\omega_{n+3}^{t}(e_i))^2\}\\ +\mathrm{trace}(A_{n+2}^2)+2\,\mathrm{trace}(A_{n+2}A_{n+3})+\mathrm{trace}(A_{n+3}^2)=0.\end{aligned}\tag{4.7}$$

Let $\xi = e_{n+2}+e_{n+3}$. Then (4.7) yields

$$\sum_{i=1}^{n}\sum_{t=n+4}^{m}\{\omega_{n+2}^{t}(e_i)+\omega_{n+3}^{t}(e_i)\}^2+\mathrm{trace}(A_\xi^2)=0.$$

Consequently, we have

$$A_\xi=0 \quad\text{and}\quad \langle D\xi,e_t\rangle=\langle D(e_{n+2}+e_{n+3}),e_t\rangle=0,\quad t=n+4,\ldots,m.\tag{4.8}$$

Without loss of generality we may assume $\omega_{n+2}^{n+3}=0$. This can be seen as follows. Since $A_\xi=0$, we have $A_{n+2}=-A_{n+3}$. Hence, by Ricci's equation, we have

$$\langle R^D(X,Y)e_{n+2},e_{n+3}\rangle=\langle[A_{n+2},A_{n+3}]X,Y\rangle=0$$

which implies $d\omega_{n+2}^{n+3}=0$. Hence, according to Poincare's lemma,

$$\omega_{n+2}^{n+3}=-d\theta\tag{4.9}$$

locally for some function θ. Put

$$\bar e_{n+2}=\sinh\theta e_{n+2}+\cosh\theta e_{n+3},\quad \bar e_{n+3}=\cosh\theta e_{n+2}+\sinh\theta e_{n+3}.\tag{4.10}$$

Then $\{\bar e_{n+2},\bar e_{n+3}\}$ is orthonormal such that $\bar e_{n+2}$ is time-like and e_{n+3} is space-like, and also H is parallel to $\bar e_{n+2}+\bar e_{n+3}$. Using (4.9) and (4.10), it follows that $\langle D\bar e_{n+2},\bar e_{n+3}\rangle=0$. This means that we may assume $\omega_{n+2}^{n+3}=0$ by using a suitable e_{n+2},e_{n+3}. Combining this with (4.8), we get $A_\xi=0$ and $D\xi=0$. Therefore, ξ is a nonzero constant vector in E_1^m. Since ξ is light-like, so up to rigid motions of E_1^m we may put

$$\xi=(a,a,0,\ldots,0)\tag{4.11}$$

for some nonzero constant a. Combining this with (4.4), we have

$$H' = (h, h, 0, \ldots, 0), \tag{4.12}$$

for some nonzero function h on M. Since $\langle x, H'\rangle = 0$, x takes the following form:

$$x = (h + f, h + f, x_3, \ldots, x_m), \tag{4.13}$$

for some functions $f, x_3, \ldots, x_m$. Combining (2.2) and (4.13) we get

$$H = H' - x = -(f, f, x_3, \ldots, x_m). \tag{4.14}$$

From (1.4), (4.13) and (4.14), we find

$$\Delta x_3 = nx_3, \ldots, \Delta x_m = nx_m, \quad \Delta h + \Delta f = nf. \tag{4.15}$$

On the other hand, because $\Delta H = nH$, (4.14) yields $\Delta f = nf$. Therefore, by (4.15), we obtain $\Delta h = 0$, i.e., h is a harmonic function on M.

If we put $y = (x_3, \ldots, x_m)$, then, by $\langle x, x\rangle = 1$, (4.13) implies $\langle y, y\rangle = 1$, where $\langle\ ,\ \rangle$ denotes the Euclidean inner product of E^{m-2}. Therefore, y: $M \to E^{m-2}$ is a mapping from M into the unit hypersphere $S^{m-3}(1)$ of E^{m-2}. Moreover, since x is an isometric immersion, (1.1) and (4.13) imply that y is also an isometric immersion. Furthermore, since $\Delta y = ny$ by (4.15), the isometric immersion y: $M \to S^{m-1}(1)$ is a minimal immersion.

Conversely, suppose y: $M \to S^{m-3}(1) \subset E^{m-2}$ is an isometric minimal immersion from an n-dimensional Riemannian manifold M into the unit hypersphere $S^{m-3}(1)$. Put $y = (x_3, \ldots, x_m)$. Let h be a harmonic function on M and f an eigenfunction of Δ with eigenvalue n. (The existence of such an eigenfunction f is guaranteed, since M admits an isometric minimal immersion into $S^{m-3}(1)$.) If we define an immersion x: $M \to E^m_1$ by

$$x = (h + f, h + f, x_3, \ldots, x_m), \tag{4.16}$$

then, by direct computation, one may verify that x is an isometric immersion from M into the de Sitter space-time $S^{m-1}_1(1)$ whose mean curvature vector H in E^m_1 is an eigenvector of Δ with eigenvalue n.

Finally, recall that when M is a minimal submanifold of $S^{m-1}_1(1)$, we have $\Delta H = nH$ as mentioned in §1. ∎

Remark 2: A submanifold given by (4.1) is of null 2-type if h is a non-constant harmonic function on M.

5. Submanifolds satisfying $\Delta H = \lambda H$ with $\lambda > n$

First we give the following

THEOREM 5: *Let λ be a real number $> n$, c a space-like vector in E_1^m satisfying $\langle c, c\rangle = 1 - n/\lambda > 0$, and $N_1(c)$ the hypersurface of the de Sitter space-time S_1^{m-1} defined by*

$$N_1(c) = \{x \in S_1^{m-1}(1)\colon \langle c, x\rangle = \langle c, c\rangle\}. \tag{5.1}$$

Then

(a) *$N_1(c)$ is a totally umbilical hypersurface of $S_1^{m-1}(1)$ with nonzero constant mean curvature;*

(b) *if M is an n-dimensional, space-like, minimal submanifold of $N_1(c)$, then*

(b-1) *M is a pseudo-umbilical submanifold of $S_1^{m-1}(1)$;*

(b-2) *the mean curvature vector H' of M in $S_1^{m-1}(1)$ is a nonzero parallel normal vector field; and*

(b-3) *the mean curvature vector H of M in E_1^m satisfies $\Delta H = \lambda H$ for some constant $\lambda > n$.*

Conversely, if $x\colon M \to S_1^{m-1}(1) \subset E_1^m$ is an isometric immersion of an n-dimensional Riemannian manifold such that the mean curvature vector field H of M in E_1^m satisfies conditions (b-2) *and* (b-3)*, then $x(M)$ is contained in $N_1(c)$ as a minimal submanifold, where c is a space-like vector in E_1^m with $\langle c, c\rangle = 1 - n/\lambda$.*

Proof: Let $\lambda > n$, c a space-like vector in E_1^m satisfying $\langle c, c\rangle = 1 - n/\lambda > 0$, and $N_1(c)$ the hypersurface of $S_1^{m-1}(1)$ defined by (5.1). Then $N_1(c)$ is the intersection of $S_1^{m-1}(1)$ and $S_1^{m-1}(c, \sqrt{n/\lambda})$. Let $\eta = c - (1 - n/\lambda)x$. Then η is a space-like nonzero normal vector field of N in $S_1^{m-1}(1)$ with$\langle \eta, \eta\rangle = n(\lambda - n)/\lambda^2 > 0$.

From the definition of η we have $A_\eta = (1 - n/\lambda)I$. Thus, $N_1(c)$ is a non-totally geodesic, totally umbilical hypersurface of $S_1^{m-1}(1)$ with constant mean curvature. This proves statement (a). Let M be an n-dimensional space-like minimal submanifold of $N_1(c)$. Then M is a pseudo-umbilical submanifold with parallel nonzero mean curvature vector H' in $S_1^{m-1}(1)$. From these we may obtain $\Delta H = \lambda H$ with $\lambda = n(\langle H', H'\rangle + 1)$. Because H' is space-like, $\lambda > n$. This proves statement (b).

Conversely, assume $x\colon M \to S_1^{m-1}(1) \subset E_1^m$ is an isometric immersion of an n-dimensional Riemannian manifold such that the mean curvature vector field H of M in E_1^m satisfies conditions (b-2) and (b-3). Then, by (2.2)–(2.8), we may obtain, as in §3, the following formulas:

$$\langle H', H'\rangle = \frac{\lambda}{n} - 1, \tag{5.2}$$

$$n\,\mathrm{trace}(A_{n+2}^2) - (\mathrm{trace}\, A_{n+2})^2 = -n\,||De_{n+2}||^2. \tag{5.3}$$

Because the mean curvature vector H' is assumed to be parallel by (b-2), (3.8) and (5.2) imply that M is a pseudo-umbilical submanifold of $S_1^{m-1}(1)$. Therefore, by (b-2) and (b-3), we get

$$A_H = \left(\frac{\lambda}{n}\right) I, \quad DH = 0. \tag{5.4}$$

Let

$$c = x + \frac{n}{\lambda} H.$$

Then c is a space-like vector in E_1^m satisfying $\langle c, c\rangle = 1 - n/\lambda$ and $\langle x - c, x - c\rangle = n/\lambda$. Because the mean curvature vector H of M in E_1^m is parallel to $x - c$, M is a minimal submanifold of the de Sitter space-time $S_1^{m-1}(c, \sqrt{n/\lambda})$. Moreover, since $N_1(c)$ is the intersection of $S_1^{m-1}(1)$ and $S_1^{m-1}(c, \sqrt{n/\lambda})$, M is a minimal submanifold of $N_1(c)$. ∎

Remark 3: If M is a space-like hypersurface of the de Sitter space-time $S_1^{n+1}(1)$ satisfying the condition $\Delta H = \lambda H$, then condition (b-2) of Theorem 5 holds automatically. Thus, by Theorems 2, 4 and 5, we have complete classification of such hypersurfaces.

Theorem 5 provides us many examples of space-like submanifolds of $S_1^{m-1}(1)$ satisfying the condition: $\Delta H = \lambda H$ with $\lambda > n$. The following theorem provides us many other examples.

THEOREM 6: *Let $m \geq 6$, r be a real number such that $0 < r < 1$, and*

$$y = (y_4, \ldots, y_m)\colon M \to S^{m-4}(r) \subset E^{m-3}$$

an isometric minimal immersion from an n-dimensional Riemannian manifold M into the hypersphere $S^{m-4}(r)$ of E^{m-3} centered at the origin and with radius r.

Then, for any non-constant harmonic function h on M and any eigenfunction f of Δ on M with eigenvalue n/r, the mapping $x\colon M \to E_1^m$ given by

$$x = (f + h, f + h, \sqrt{1 - r^2}, y_4, \dots, y_m) \tag{5.5}$$

defines an isometric immersion from M into the de Sitter space-time $S_1^{m-1}(1)$ satisfying the following two conditions:

(a) $\Delta H = \lambda H$ *with* $\lambda = n/r^2 > n$;

(b) $DH = \omega\xi$ *for some 1-form* $\omega \neq 0$ *and constant light-like vector* $\xi \neq 0$.

Conversely, if $x\colon M \to S_1^{m-1}(1) \subset E_1^m$ is an isometric immersion of an n-dimensional Riemannian manifold satisfying conditions (a) *and* (b), *then, up to rigid motions of E_1^m, the immersion x is given by* (5.5) *for some isometric minimal immersion $y = (y_4, \dots, y_m)\colon M \to S^{m-4}(r)$ ($\subset E^{m-3}$), real number r with $0 < r < 1$, harmonic function h, and eigenfunction f of Δ with eigenvalue n/r^2.*

Proof: Let r be a real number such that $0 < r < 1$ and let

$$y = (y_4, \dots, y_m)\colon M \to S^{m-4}(r) \subset E^{m-3}$$

be an isometric minimal immersion from an n-dimensional Riemannian manifold M into the hypersphere $S^{m-4}(r)$ of E^{m-3} centered at the origin and with radius r. Then, for any harmonic function h on M and any eigenfunction f of Δ on M with eigenvalue n/r^2, we define a mapping $x\colon M \to E_1^m$ by

$$x = (f + h, f + h, \sqrt{1 - r^2}, y_4, \dots, y_m). \tag{5.6}$$

From (5.6) we obtain $dx = (df + dh, df + dh, 0, dy_4, \dots, dy_m)$. Hence, by (1.1), $\langle dx, dx\rangle = \langle dy, dy\rangle$. This implies that the mapping x defined by (5.6) is an isometric immersion from M into E_1^m. Furthermore, (1.1) and (5.6) yield $1 = \langle x, x\rangle = 1 - r^2 + \langle y, y\rangle$. Thus, $x(M)$ is contained in the de Sitter space-time $S_1^{m-1}(1)$.

On the other hand, because $y = (y_4, \dots, y_m)\colon M \to S^{m-4}(r) \subset E^{m-3}$ is an isometric minimal immersion from the n-dimensional Riemannian manifold M into the hypersphere $S^{m-4}(r)$ of E^{m-3} centered at the origin and with radius r, we have $\Delta y = (n/r^2)y$. Therefore, (1.4) and (5.6) yield

$$H = -\frac{1}{r^2}(f, f, 0, y_4, \dots, y_m), \tag{5.7}$$

by the fact that h is a harmonic function. From (5.7) we obtain $\Delta H = \lambda H$ with $\lambda = n/r^2 > n$. Moreover, (2.2) and (5.7) give

$$H' = \frac{\lambda}{n}(h, h, \sqrt{1-r^2}, 0, \ldots, 0) + \left(1 - \frac{\lambda}{n}\right) x. \tag{5.8}$$

Therefore

$$A_{H'} = \left(\frac{\lambda}{n} - 1\right) I, \quad D_X H' = \frac{\lambda}{n}(Xh, Xh, 0, \ldots, 0), \tag{5.9}$$

for any vector X tangent to M. We put

$$\xi = (1, 1, 0, \ldots, 0), \quad \omega = \left(\frac{\lambda}{n}\right) dh. \tag{5.10}$$

Then ξ is a nonzero light-like constant vector in E_1^m and ω is a nonzero 1-form on M satisfying $DH = DH' = \omega\xi$.

Conversely, suppose $x\colon M \to S_1^{m-1}(1) \subset E_1^m$ is an isometric immersion of an n-dimensional Riemannian manifold satisfying conditions (a) and (b) of Theorem 6. Then, (2.2)–(2.8) and condition (a) imply

$$\langle H', H' \rangle = \frac{\lambda}{n} - 1, \quad \langle H, H \rangle = \frac{\lambda}{n}, \tag{5.11}$$

$$n \operatorname{trace}(A_{n+2}^2) - (\operatorname{trace} A_{n+2})^2 = -n \, ||De_{n+2}||^2. \tag{5.12}$$

From (5.11) we see that H' is a space-like nonzero vector.

We put $H' = \alpha' e_{n+2}$. Condition (b) yields

$$||De_{n+2}||^2 = \sum_{i=1}^{n} < D_{e_i} e_{n+2}, D_{e_i} e_{n+2} >= 0. \tag{5.13}$$

Combining (3.8), (5.12) and (5.13), it follows that M is a pseudo-umbilical submanifold of $S_1^{m-1}(1)$ with

$$A_{H'} = \left(\frac{\lambda}{n} - 1\right) I, \quad DH' = \omega\xi. \tag{5.14}$$

Therefore, by applying the equation of Weingarten, we find

$$\tilde{\nabla}_X H' = \left(1 - \frac{\lambda}{n}\right) X + \omega(X)\xi. \tag{5.15}$$

On the other hand, since $\tilde{\nabla}_X x = X$ for any vector X tangent to M, (5.15) yields

$$\tilde{\nabla}_X \left(H' + \left(\frac{\lambda}{n} - 1 \right) x \right) = \omega(X)\xi. \tag{5.16}$$

Because ξ is a nonzero light-like constant vector in E_1^m, up to rigid motions of E_1^m, ξ is given by $\xi = (\mu, \mu, 0, \ldots, 0)$ for some nonzero function μ. Furthermore, replacing ω by $\mu\omega$, if necessary, we may assume

$$\xi = (1, 1, 0, \ldots, 0). \tag{5.17}$$

From (5.16) and (5.17) we get

$$d\left(H' + \left(\frac{\lambda}{n} - 1 \right) x \right) = (\omega, \omega, 0, \ldots, 0). \tag{5.18}$$

Put

$$H' = (f_1, f_2, \ldots, f_m), \quad x = (x_1, \ldots, x_m). \tag{5.19}$$

Then (5.18) and (5.19) imply

$$\omega = dh, \quad h = f_1 + \left(\frac{\lambda}{n} - 1 \right) x_1, \quad dh = d\left(f_2 + \left(\frac{\lambda}{n} - 1 \right) x_2 \right). \tag{5.20}$$

Combining (2.2), (5.18), (5.19) and (5.20), we find

$$H' + \left(\frac{\lambda}{n} - 1 \right) x = (h, h, 0, \ldots, 0) + c, \tag{5.21}$$

for some $c = (c_1, \ldots, c_m) \in E_1^m$. From (5.11) and (5.21) it follows that

$$2\langle c, (h, h, 0, \ldots, 0) \rangle = \frac{\lambda}{n} \left(\frac{\lambda}{n} - 1 \right) - \langle c, c \rangle. \tag{5.22}$$

(1.1) and (5.22) imply that $(c_1 - c_2)h$ is a constant function on M. Because $\omega = dh$ is a nonzero 1-form, h is not a constant function. Hence, $c_1 = c_2$. If we denote c_1 by b and replace $h + b$ by h, then we have $c = (0, 0, c_3, \ldots, c_m)$. Therefore, by applying a rigid motion of E_1^m if necessary, we can assume c takes the form $c = (0, 0, e, 0, \ldots, 0)$. Therefore, by (2.2), (5.19), and (5.21), we have

$$H + \frac{\lambda}{n} x = H' + \left(\frac{\lambda}{n} - 1 \right) x = (h, h, e, 0, \ldots, 0), \quad e = \sqrt{\frac{\lambda}{n} \left(\frac{\lambda}{n} - 1 \right)}. \tag{5.23}$$

By applying condition (a), (1.4) and (5.23), we obtain

$$0 = \Delta H - \lambda H = \Delta H + \frac{\lambda}{n}\Delta x = (\Delta h, \Delta h, 0, \dots, 0).$$

Therefore, $\Delta h = 0$. Consequently, h is a non-constant harmonic function on M. Combining (2.2) and (5.23) we get

$$x = \frac{n}{\lambda}((h, h, e, 0, \dots, 0) - H), \quad \Delta h = 0. \tag{5.24}$$

By using $\langle x, x\rangle = 1$ together with (5.11) and (5.24), we may obtain

$$\langle H, (h, h, e, 0, \dots, 0)\rangle = 0.$$

Therefore, by (1.1) and (5.19),

$$ef_3 = (f_1 - f_2)h. \tag{5.25}$$

Moreover, from condition (a) and (5.19),

$$\Delta f_A = \lambda f_A, \quad \lambda = \frac{n}{r^2}, \quad A = 1, \dots, m. \tag{5.26}$$

From (2.2), (2.4) and (5.14) it follows $A_H = \lambda/nI$, $DH = \omega\xi$. Therefore, by applying the equation of Weingarten and (5.17), we find

$$(Xf_1, \dots, Xf_m) = \tilde{\nabla}_X H = -\left(\frac{\lambda}{n}\right) X + (\omega(X), \omega(X), 0, \dots, 0). \tag{5.27}$$

By taking the inner product of (5.26) with the light-like normal vector ξ, we get $Xf_1 - Xf_2 = 0$ for any vector X tangent to M. This shows that $f_1 - f_2$ is a constant function. Combining this with (5.24), (5.25) and (5.26), we obtain $f_1 = f_2$, $f_3 = 0$. Put

$$f = \frac{n}{\lambda}f_1, \quad y_A = -\frac{n}{\lambda}f_A, \quad A = 4, \dots, m \quad \text{and} \quad h_1 = \frac{n}{\lambda}h.$$

Then

$$x = (f + h_1, f + h_1, \sqrt{1 - n/\lambda}, y_4, \dots, y_m), \tag{5.28}$$

where h_1 is a nonconstant harmonic function, and $f, y_4, \dots, y_m$ are eigenfunctions of Δ with eigenvalue n/r^2. Put $y = (y_4, \dots, y_m)$. Then $\langle dy, dy\rangle = \langle dx, dx\rangle$, which implies that y is an isometric immersion from M into E^{m-3}. Moreover, because $1 = \langle x, x\rangle = 1 - n/\lambda + \langle y, y\rangle$, we have $\langle y, y\rangle = r^2$. Thus, $y(M)$ is contained in the hypersphere $S^{m-4}(r)$ of E^{m-3} centered at the origin and with radius $r = \sqrt{n/\lambda}$. Furthermore, because $y_4, \dots, y_m$ are eigenfunctions of Δ with eigenvalue n/r, M is immersed in $S^{m-4}(r)$ as a minimal submanifold. ∎

Thereom 6 shows in particular that, for $m \geq 6$, there exist many space-like surfaces in $S_1^{m-1}(1)$ whose mean curvature vector H in E_1^m satisfies the condition $\Delta H = \lambda H$ for some $\lambda > 2$ and $DH \neq 0$. However, the following result shows that there do not exist such surfaces if $m \leq 5$.

THEOREM 7: *Let x: $M \to S_1^4(1) \subset E_1^5$ be an isometric immersion from a 2-dimensional Riemannain manifold M into the 4-dimensional de Sitter space-time $S_1^4(1)$ such that the mean curvature vector H of M in E_1^5 satisfies the condition $\Delta H = \lambda H$ for some $\lambda > 2$. Then the mean curvature vector H is parallel in the normal bundle and M lies in a non-totally geodesic, totally umbilical hypersurface of $S_1^4(1)$ as a minimal submanifold.*

Proof: Under the hypothesis, the mean curvature vector H' of M in $S_1^4(1)$ is a space-like vector field. We choose orthonormal normal frame field e_3, e_4, e_5 such that $e_3 = x$ and $H' = \alpha' e_4$ with $\alpha' = |H'|$. We have $\epsilon_3 = \epsilon_4 = -\epsilon_5 = 1$. By using (2.2), (2.3) and (2.8), we may find

$$DH = DH' = \omega_4^5 e_5, \tag{5.29}$$

$$\langle H', H'\rangle = \frac{1}{2}(\lambda - 2), \quad \operatorname{trace} A_{DH} = 0, \quad \operatorname{trace}(\nabla\omega_4^5) = 0, \tag{5.30}$$

$$2 + \operatorname{trace}(A_4^2) - ||\omega_4^5||^2 = \lambda. \tag{5.31}$$

If $DH = 0$, then $DH' = 0$; and hence, the result follows from Theorem 4. So we assume $DH = \omega_4^5 e_5 \neq 0$. In this case, $\omega_4^5 \neq 0$. Since $\operatorname{trace} A_5 = 0$, (5.29) and $\operatorname{trace} A_{DH} = 0$ yield $A_5 = 0$. Choose e_1, e_2 to be the eigenvectors of A_4 with eigenvalues κ_1, κ_2, respectively. By applying the equation of Codazzi and the condition $A_5 = 0$, we may obtain

$$\kappa_1\omega_4^5(e_2) = \kappa_2\omega_4^5(e_1) = 0, \tag{5.32}$$

$$(5.33)\qquad e_1\kappa_2 = \omega_1^2(e_2)(\kappa_1 - \kappa_2), \quad e_2\kappa_1 = \omega_2^1(e_1)(\kappa_2 - \kappa_1).$$

Without loss of generality, we may assume by (5.32) that $\kappa_2 = \omega_4^5(e_2) = 0$. Since the mean curvature of M in $S_1^4(1)$ is constant and $\kappa_2 = 0$, κ_1 is a nonzero constant. Therefore, (5.33) implies $\omega_1^2 = 0$. Therefore, M is flat.

On the other hand, because $\det A_3 = 1$ and $\det A_4 = \det A_5 = 0$, the equation of Gauss implies that M has constant Gaussian curvature 1, so M is non-flat. Thus, we obtain a contradiction. ∎

Remark 4: For submanifolds in a hypersphere of a Euclidean space and submanifolds of a hyperbolic space, we may prove (1): *If M is an n-dimensional submanifold of a unit hypersphere $S^{m-1}(1)$ of a Euclidean m-space E^m, then M is a minimal submanifold of $S^{m-1}(1)$ if and only if the mean curvature vector H of M in E^m is an eigenvector of Δ with eigenvalue n;* and (2) *If M is an n-dimensional submanifold of the hyperbolic space $H^{m-1}(-1)$, imbedded standardly in the Minkowski space-time E_1^m, then M is a minimal submanifold of $H^{m-1}(-1)$ if and only if the mean curvature vector of M in E_1^m is an eigenvector of Δ with eigenvalue $-n$.*

Remark 5: In [6], Dillen, Pas and Verstraelen studied surfaces in E^3 whose Gauss map G satisfies the condition: $\Delta G = AG$ for some 3×3 matrix A. Investigation of submanifolds in an m-dimensional Euclidean space (or pseudo-Euclidean space) whose mean curvature vector H satisfies the condition: $\Delta H = AH$ for some $m \times m$ matrix A can be found in [5].

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